Chapter 2:
Matrix Algebra

Sec. 2.4: Matrix Inverses

Matrix Inverses

Definition 2.11 Matrix Inverses

If A is a square matrix, a matrix B is called an ***inverse*** *of A if and only if*

$$AB = I \quad \text{and} \quad BA = I$$

A matrix A that has an inverse is called an ***invertible matrix***.[8]

Matrix Inverses

Ex 1: Show that $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Matrix Inverses

Ex 2: Show that $A = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ has no inverse

Matrix Inverses

Theorem 2.4.1

If B and C are both inverses of A, then $B = C$.

Matrix Inverses

<u>Ex 3</u>: If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, show that $A^3 = I$ and so find A^{-1}

Matrix Inverses

Formula for the inverse of a 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define $\det A = ad - bc$ and $adj\ A = \begin{bmatrix} d & -b \\ -c & -a \end{bmatrix}$

Then A^{-1} only exists if $\det A \neq 0$. In this case, $A^{-1} = \frac{1}{\det A}\ adj\ A$.

Ex 4: Find the inverse of the following 2×2 matrices…

a) $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Matrix Inverses

Ex 5: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, show that A has an inverse if and only if $\det A \neq 0$, and in this case

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Inverses and Linear Systems

Theorem 2.4.2

Suppose a system of n equations in n variables is written in matrix form as

$$A\boldsymbol{x} = \boldsymbol{b}$$

If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Inverses and Linear Systems

Ex 6: Use matrix inverses to solve the system of equations

$$\begin{aligned} 5x_1 - 3x_2 &= -4 \\ 7x_1 + 4x_2 &= 8 \end{aligned}$$

Finding the Inverse of a Matrix

Matrix Inversion Algorithm

If A is an invertible (square) matrix, there exists a sequence of elementary row operations that carry A to the identity matrix I of the same size, written $A \to I$. This same series of row operations carries I to A^{-1}; that is, $I \to A^{-1}$. The algorithm can be summarized as follows:

$$\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

where the row operations on A and I are carried out simultaneously.

Theorem 2.4.3

If A is an $n \times n$ matrix, either A can be reduced to I by elementary row operations or it cannot. In the first case, the algorithm produces A^{-1}; in the second case, A^{-1} does not exist.

Finding the Inverse of a Matrix

Ex 7: Use the inversion algorithm to find the inverse of

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Finding the Inverse of a Matrix

<u>Ex 7</u>: Use the inversion algorithm to find the inverse of

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

Properties of Inverses

Cancellation Laws

Let A be an invertible matrix. Show that:

1. If $AB = AC$, then $B = C$.
2. If $BA = CA$, then $B = C$.

Proof:

Properties of Inverses

If A and B are invertible $n \times n$ matrices, show that their product AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

Properties of Inverses

Theorem 2.4.4

All the following matrices are square matrices of the same size.

1. I *is invertible and* $I^{-1} = I$.
2. *If* A *is invertible, so is* A^{-1}*, and* $(A^{-1})^{-1} = A$.
3. *If* A *and* B *are invertible, so is* AB*, and* $(AB)^{-1} = B^{-1}A^{-1}$.
4. *If* $A_1, A_2, \ldots, A_k$ *are all invertible, so is their product* $A_1A_2\cdots A_k$*, and*
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$
5. *If* A *is invertible, so is* A^k *for any* $k \geq 1$*, and* $(A^k)^{-1} = (A^{-1})^k$.
6. *If* A *is invertible and* $a \neq 0$ *is a number, then* aA *is invertible and* $(aA)^{-1} = \frac{1}{a}A^{-1}$.
7. *If* A *is invertible, so is its transpose* A^T*, and* $(A^T)^{-1} = (A^{-1})^T$.

Proof:

Properties of Inverses

Corollary 2.4.1

A square matrix A is invertible if and only if A^T is invertible.

Proof:

Properties of Inverses

<u>Ex 8</u>: Find A if $(A^T - 2I)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Properties of Inverses

Theorem 2.4.5: Inverse Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*
2. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *A can be carried to the identity matrix I_n by elementary row operations.*
4. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution $\mathbf{x}$ for every choice of column $\mathbf{b}$.*
5. *There exists an $n \times n$ matrix C such that $AC = I_n$.*

Proof:

Properties of Inverses

Corollary 2.4.1

If A and C are square matrices such that $AC = I$, then also $CA = I$. In particular, both A and C are invertible, $C = A^{-1}$, and $A = C^{-1}$.

Proof:

Properties of Inverses

Corollary 2.4.2

An $n \times n$ *matrix* A *is invertible if and only if* $\operatorname{rank} A = n$.

Proof: